Beyond Isomorphic Holomorphs

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Holomorphs and Isomorphic Holomorphs

For G a (finite) group, the abstract holomorph is the semi-direct product

$$\mathrm{Hol}(G) = G \rtimes Aut(G)$$

whose study does not specifically necessitate the discussion of regular permutation groups.

However, as we are all quite interested in regular permutation groups, due to their centrality in the study of Hopf-Galois structures and braces we shall work within the setting of left regular representations and regularity in general.

In terms of permutations, for a general G embedded as $\lambda(G)$ in $B = Perm(G)$ we can identify

$$\mathrm{Hol}(G) = \mathrm{Norm}_{\mathrm{B}}(\lambda(G)) = \rho(G)A_z$$

where A_z is any stabilizer of a point in $z \in G$.

Typically we choose the stabilizer of the identity element, as this is most naturally identified with $Aut(G)$, although all such point stabilizers are conjugate.

We begin by reminding the audience of a *very* classical result regarding groups whose holomorphs are isomorphic.

Consider the dihedral and quaternionic groups of order $4n$ for $n \geq 3$.

$$D_{2n} = \langle x, t \mid x^{2n} = 1, t^2 = 1, xt = tx^{-1} \rangle$$
$$Q_n = \langle x, t \mid x^{2n} = 1, t^2 = x^n, xt = tx^{-1} \rangle$$

which are not isomorphic, but do have certain structural similarities.

And by utilizing the same symbols 'x' and 't' to represent their generators, we observe that as sets, they contain same elements:

$$\{t^a x^b \mid a \in \mathbb{Z}_2, b \in \mathbb{Z}_{2n}\}$$

Again, these are, of course, not isomorphic, but, viewed as subgroups of $Perm(\{t^a x^b\})$, we can show that they have a common automorphism group:

$$
\begin{aligned}
A &= \{\phi_{(i,j)} \mid i \in \mathbb{Z}_{2n}, j \in U(\mathbb{Z}_{2n})\} \\
&\cong \mathbb{Z}_{2n} \rtimes U(\mathbb{Z}_{2n}) \\
&\cong \mathrm{Hol}(\mathbb{Z}_{2n}) \\
&\text{where } \phi_{(i,j)}(t^a x^b) = t^{ia} x^{ia+jb}
\end{aligned}
$$

Beyond this, one can show that not only do we have an isomorphism

$$\mathrm{Hol}(D_{2n}) \cong \mathrm{Hol}(Q_n)$$

as abstract groups, but also, by viewing both as permutations on $\{t^a x^b\}$ we can show that $\mathrm{Hol}(D_{2n}) = \mathrm{Hol}(Q_n)$ as subgroups of $Perm(\{t^a x^b\})$ since one can show that:

$$\rho_Q(x^b)\phi_{(i,j)} = \rho_D(x^b)\phi_{(i,j)}$$
$$\rho_Q(tx^b)\phi_{(i,j)} = \rho_D(tx^{b+n})\phi_{(i+n,j)}$$

So in particular

$$\{\rho_D(x^b), \rho_D(tx^{b+n})\phi_{(n,1)} \mid b \in \mathbb{Z}_{2n}\}$$

is a normal, **regular** subgroup of $\mathrm{Hol}(D_{2n})$ that is isomorphic to Q_n.

The significance of this will be seen a bit later.

In this family of groups, there is a notable exception to this equivalence, namely when $n = 2$, corresponding to the groups Q_2 and D_4 of order 8, where the difference is most obvious by considering the sizes of the respective automorphism groups:

$$Aut(D_4) \cong D_4$$
$$Aut(Q_2) \cong S_4$$

which therefore automatically precludes their holomorphs from being isomorphic.

We shall, however, be able to demonstrate a 'higher' equivalence between D_4 and Q_2 which broadly generalizes the symmetry that exists between the holomorphs of the other dihedral and quaternionic groups.

Looking at Hopf-Galois structures/braces for a moment, there are interesting implications for two abstract groups having isomorphic holomorphs.

In particular there is a symmetry between the Hopf-Galois structures of types corresponding to isomorphism classes of groups with isomorphic holomorphs.

Isomorphic Holomorphs and Hopf-Galois Structures

For a Galois extension L/K where $G = Gal(L/K)$ any Hopf-Galois structure corresponds to a regular subgroup $N \leq B = \mathrm{Perm}(G)$ normalized by $\lambda(G)$, so in particular

$$\lambda(G) \leq \mathrm{Norm_B}(N)$$

where $\mathrm{Norm_B}(N)$ is isomorphic to $\mathrm{Hol}(N)$, where N need not be isomorphic to G, although $|G| = |N|$ by regularity.

Let $R(G, [M])$ be the set of all such regular N which are isomorphic to M (i.e. of 'type' M) where M is an isomorphism class of group of size $|G|$.

If for two distinct isomorphism classes $[M_1]$ and $[M_2]$ of the same size we have that

$$\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2) \text{ (as abstract groups)}$$

and **if** in $\mathrm{Hol}(M_1) = \mathrm{Norm}_B(\lambda(M_1))$ there exists a normal regular subgroup isomorphic to M_2 then one has a bijection

$$R(G, [M_1]) \longleftrightarrow R(G, [M_2])$$

The question we are faced with is that, assuming $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ as abstract groups, isn't it automatically true that (as permutation groups) $\mathrm{Hol}(M_1)$ contain a regular normal subgroup isomorphic to M_2?

After all, $M_2 \triangleleft \mathrm{Hol}(M_2)$ since $\mathrm{Hol}(M_2)$ is an extension of M_2 by $Aut(M_2)$, so if $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ then there exists at least one subgroup $\hat{M}_2 \triangleleft \mathrm{Hol}(M_1)$ where $\hat{M}_2 \cong M_2$.

The issue is that there are examples of groups M_1, M_2 of the same order, where $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ but where $\mathrm{Hol}(M_1)$ contains *non-regular* normal subgroups isomorphic to M_2.

For those examples I've computed however, when $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ there does indeed always exist a regular normal subgroup of $\mathrm{Hol}(M_1)$ that is isomorphic to M_2, but I don't have a proof that this holds in general.

A Tale of Two Groups

Again, this presentation is about regular permutation groups, but more fundamentally about groups, in particular collections of groups which *should* have isomorphic holomorphs, or be equivalent when viewed in some larger setting.

One motivating example is the following pair of groups of order 16:

$$C_4 \times C_4 = \langle x, y \mid x^4 = 1, y^4 = 1, [x, y] = 1 \rangle$$
$$C_4 \rtimes C_4 = \langle z, w \mid z^4 = 1, w^4 = 1, wzw^{-1} = z^{-1} \rangle$$

which are obviously not isomorphic, but which have similarities, such as having the same distribution of element orders.

We can ask if their automorphism groups and holomorphs are isomorphic.

Alas, the answer is no:

$$|Aut(C_4 \times C_4)| = 1536$$
$$|Aut(C_4 \rtimes C_4)| = 512$$

so obviously their holomorphs aren't isomorphic either.

But, as we mentioned earlier, there is a higher equivalence we can demonstrate, but we first need to discuss the appropriate generalizations of the holomorph.

And this requires us to explore regular permutations groups in some generality.

Classes of Regular Permutation Groups

For a group G of a certain size, embedded in its group of permutations $B = Perm(G)$ via the left regular representation $\lambda(G)$ we can enumerate the classes of regular subgroups $N \leq B$ that are normalized by $\lambda(G)$.

We may also, in parallel, consider the totality of those regular subgroups of $\mathrm{Hol}(G) = \mathrm{Norm}_{\mathrm{B}}(\lambda(G))$.

In particular, we consider those regular subgroups of B that are normalized by $\lambda(G)$ and those that normalize $\lambda(G)$.

The regularity condition, of course, does not necessarily imply that any regular N normalized by $\lambda(G)$ (or normalizing $\lambda(G)$) is isomorphic to G.

However, in the interest of succinctness, we shall restrict our attention to those regular N which are isomorphic to G.

For $B = Perm(G)$ we define

$$\mathcal{R}(G) = \{N \leq B \mid N \text{ is regular, } N \cong G,\ \lambda(G) \leq \mathrm{Norm_B}(N)\}$$
$$\mathcal{S}(G) = \{M \leq \mathrm{Hol}(G) \mid M \text{ is regular and } M \cong G\}$$

We recall an important symmetry between $\mathcal{S}(G)$ and $\mathcal{R}(G)$.

The fact that these N are regular, and isomorphic to $\lambda(G)$ implies that each such N is a conjugate of $\lambda(G)$ by some element $\gamma \in B$.

And so if $N \in \mathcal{S}(G)$ or $\mathcal{R}(G)$, then $N = \gamma\lambda(G)\gamma^{-1}$, and moreover, if we let $\hat{\gamma} = \gamma h$ for any element $h \in \mathrm{Hol}(G)$ then clearly

$$\hat{\gamma}\lambda(G)\hat{\gamma}^{-1} = \gamma h\lambda(G)h^{-1}\gamma^{-1} = \gamma\lambda(G)\gamma^{-1} = N$$

and thus each element of the coset $\gamma\,\mathrm{Hol}(G)$ corresponds to the same element of $\mathcal{S}(G)$ or $\mathcal{R}(G)$.

The symmetry between $\mathcal{R}(G)$ and $\mathcal{S}(G)$ comes from the fact that for a given $\gamma \in B$:

$$\gamma\lambda(G)\gamma^{-1} \in \mathcal{S}(G) \longleftrightarrow \gamma^{-1}\lambda(G)\gamma \in \mathcal{R}(G)$$

and thus $\mathcal{R}(G)$ and $\mathcal{S}(G)$ are in bijective correspondence with each other.

However, one should not generally expect them to be equal, although they do overlap to some degree, and this overlap is important for our analysis.

The set

$$\mathcal{S}(G) \cap \mathcal{R}(G)$$

consists of those regular subgroups $N \leq Perm(G)$ (where $N \cong G$) with the property that they normalize, and are normalized by, $\lambda(G)$.

As such, there exists a set of 'parameters'

$$\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \ldots, \beta_t\}$$

and thus a set of distinct coset representatives of $\mathrm{Hol}(G)$ with the property that

$$\mathcal{S}(G) \cap \mathcal{R}(G) = \{\beta_i \lambda(G) \beta_i^{-1}\}$$

where if $\beta\lambda(G)\beta^{-1} \in \mathcal{S}(G) \cap \mathcal{R}(G)$ then $\beta^{-1}\lambda(G)\beta \in \mathcal{S}(G) \cap \mathcal{R}(G)$.

Since any $\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \ldots, \beta_t\}$ gives rise to a set of distinct cosets $\beta_i \operatorname{Hol}(G)$, a natural question to ask is whether these cosets (or even the coset representatives) form a group?

There is an important subset of $\mathcal{S}(G) \cap \mathcal{R}(G)$ for which it does.

Definition

For G a group, embedded in $B = Perm(G)$ as $\lambda(G)$, define

$$\begin{aligned}\mathcal{H}(G) &= \{N \leq B \mid N \text{ regular },\ N \cong G,\ \mathrm{Norm}_{\mathrm{B}}(N) = \mathrm{Hol}(G)\}\\ &= \{N \triangleleft \mathrm{Hol}(G) \mid N \text{ regular },\ N \cong G\}\end{aligned}$$

It is clear therefore that $\mathcal{H}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$.

And since $\mathcal{H}(G)$ consists of regular subgroups isomorphic to $\lambda(G)$, they are conjugate to $\lambda(G)$, and so $\mathcal{H}(G)$ determines a set of cosets $\{\gamma_i \operatorname{Hol}(G)\}$ of $\operatorname{Hol}(G)$.

And for any such γ we have that

$$\gamma \operatorname{Norm}_{\mathrm{B}}(\lambda(G))\gamma^{-1} = \operatorname{Norm}_{\mathrm{B}}(\gamma\lambda(G)\gamma^{-1}) = \operatorname{Norm}_{\mathrm{B}}(\lambda(G))$$

and so γ normalizes $\operatorname{Hol}(G)$, that is, it's an element of the so-called *multiple holomorph* $\operatorname{NHol}(G) = \operatorname{Norm}_{\mathrm{B}}(\operatorname{Hol}(G))$.

We observe that $\operatorname{NHol}(G)$ is a group, obviously containing $\operatorname{Hol}(G)$ as a normal subgroup.

The multiple holomorph provides our first example of the higher equivalences we hinted at earlier, generalizing the property of having isomorphic holomorphs.

In this case, the groups in question are also quite similar in structure:

$$C_8 \times C_2 = \langle x, y \mid x^8 = 1, y^2 = 1, [x, y] = 1 \rangle$$
$$C_8 \rtimes C_2 = \langle z, w \mid z^8 = 1, w^2 = 1, wzw^{-1} = z^5 \rangle$$

and (like the two other groups of order 16 we've mentioned) they have the same distribution of element orders.

Moreover, they have isomorphic automorphism groups, namely

$$D_4 \times C_2$$

However, it turns out that $\mathrm{Hol}(C_8 \times C_2) \not\cong \mathrm{Hol}(C_8 \rtimes C_2)$.

For the record, $\mathrm{Hol}(C_8 \times C_2)$ is group (256,16870) in the 'SmallGroups' library in GAP and Magma, while $\mathrm{Hol}(C_8 \rtimes C_2)$ is group (256,16860).

But.. their multiple holomorphs are isomorphic.

In both cases $\mathrm{NHol}(G)$ is a split extension of $\mathrm{Hol}(G)$ by a Klein-4 group.

In case there is some doubt, here are the presentations in GAP, where we can work with generators of the groups embedded as regular subgroups of S_{16}, and in fact, we can choose the order 8 generator of each to be identical.

$G_1 = \langle x, y\rangle$ and $G_2 = \langle z, w\rangle$ where

$$
\begin{aligned}
x &= (1,2,4,7,5,8,11,14)(3,6,9,12,10,13,15,16)\\
y &= (1,3)(2,6)(4,9)(5,10)(7,12)(8,13)(11,15)(14,16)\\
z &= (1,2,4,7,5,8,11,14)(3,6,9,12,10,13,15,16)\\
w &= (1,3)(2,13)(4,9)(5,10)(6,8)(7,16)(11,15)(12,14)\\
Aut(G_1) &= \langle(2,6,8,13)(3,10)(7,12,14,16)(9,15),\\
&\quad (2,7)(3,10)(4,11)(6,16)(8,14)(12,13),\\
&\quad (2,14)(4,11)(6,16)(7,8)(9,15)(12,13)\rangle\\
Aut(G_2) &= \langle(2,6,8,13)(3,10)(4,11)(7,16,14,12),\\
&\quad (3,10)(6,13)(9,15)(12,16),\\
&\quad (2,14)(3,10)(4,11)(6,12)(7,8)(13,16)\rangle
\end{aligned}
$$

And given these generators for $\mathrm{Hol}(G_1)$ and $\mathrm{Hol}(G_2)$ we can, in fact, show that $\mathrm{NHol}(G_1)$ and $\mathrm{NHol}(G_2)$ are not just isomorphic, but are, in fact, *equal* as subgroups of S_{16}.

Namely, for

$$T = \langle (4,11)(7,14)(9,15)(12,16), (4,11)(6,13)(7,14)(9,15) \rangle$$

both $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$, are conjugates of G_1 and G_2 by the elements of this same (transversal) T which is a group isomorphic to $C_2 \times C_2$ where

$$\mathrm{NHol}(G_1) = \mathrm{Hol}(G_1)T = \mathrm{Hol}(G_2)T = \mathrm{NHol}(G_2)$$

as split extensions of their respective holomorphs.

So what about the pairs

$$C_4 \times C_4 \text{ and } C_4 \rtimes C_4$$

and

$$D_4 \text{ and } Q_2$$

which we observed do not have isomorphic holomorphs?

Are their multiple holomorphs isomorphic?

Well, for $C_4 \times C_4$ and $C_4 \rtimes C_4$ we have

$$|NHol(C_4 \times C_4)| = 1536$$
$$|NHol(C_4 \rtimes C_4)| = 4096$$

and for D_4 and Q_2 we have that

$$|NHol(D_4)| = 128$$
$$|NHol(C_4 \rtimes C_4)| = 384$$

so it is definitely not the case that their multiple holomorphs are isomorphic, let alone equal.

So is there any still 'higher' equivalence between these pairs?

In many cases, $\mathcal{S}(G) \cap \mathcal{R}(G)$ is properly larger than $\mathcal{H}(G)$, so the cosets $\{\beta_i \operatorname{Hol}(G)\}$ formed from the coset representatives

$$\pi(\mathcal{S}(G) \cap \mathcal{R}(G)) = \{\beta_1, \dots, \beta_t\}$$

need not necessarily form a group for which the orbit of $\lambda(G)$ is $\mathcal{S}(G) \cap \mathcal{R}(G)$ with $\operatorname{Hol}(G)$ being the stabilizer.

The way to recover a group structure is to select a subset of the $\{\beta_i\}$ corresponding to those members of $\mathcal{S}(G) \cap \mathcal{R}(G)$ which *mutually normalize each other*.

Going back to $\mathcal{H}(G)$ for a moment, we note that any pair $N_1, N_2 \in \mathcal{H}(G)$ normalize each other.

The reason for this is that $\mathrm{Norm}_{\mathrm{B}}(N_1) = \mathrm{Norm}_{\mathrm{B}}(N_2) = \mathrm{Hol}(G)$.

But, again, looking at the totality of $\mathcal{S}(G) \cap \mathcal{R}(G)$, there is no reason to expect its members to mutually normalize each other, so we therefore restrict to a subset for which this is guaranteed.

Definition

Let $\mathcal{Q}(G) = \bigcap_{N \in \mathcal{S}(G) \cap \mathcal{R}(G)} \{M \in \mathcal{S}(G) \cap \mathcal{R}(G) \mid N \text{ normalizes } M\}$.

We first observe the most important property of this set.

Lemma

The members of $\mathcal{Q}(G)$ mutually normalize each other.

As far as containments go, we have.

Lemma

For $\mathcal{Q}(G)$ defined above, one has $\mathcal{H}(G) \subseteq \mathcal{Q}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$.

The set $\mathcal{Q}(G)$, consisting of certain conjugates of $\lambda(G)$, gives rise to a set of parameters

$$\pi(\mathcal{Q}(G)) = \{\beta_1, \ldots, \beta_m\}$$

and therefore a set of distinct cosets of the holomorph $\{\beta_i \operatorname{Hol}(G)\}$.

In [1] we defined this generalization of the multiple holomorph.

Definition

For $\pi(\mathcal{Q}(G)) = \{\beta_1, \ldots, \beta_m\}$ which parameterizes $\mathcal{Q}(G)$, let the *quasiholomorph* of G be $\operatorname{QHol}(G) = \cup_{i=1}^{m} \beta_i \operatorname{Hol}(G)$.

Before going further, a bit of full disclosure must be made.

For any G, we have that

$$\mathcal{H}(G) \subseteq \mathcal{Q}(G) \subseteq \mathcal{S}(G) \cap \mathcal{R}(G)$$

and if $\mathcal{Q}(G) = \mathcal{H}(G)$ then $\mathrm{QHol}(G) = \mathrm{NHol}(G)$ (and is therefore obviously a well defined group) and if $\mathcal{Q}(G) = \mathcal{S}(G) \cap \mathcal{R}(G)$, then the quasiholomorph is also known to be a group.

When each of the above containments is proper then it is not (yet) known that the quasiholomorph is always closed under multiplication, but this has been demonstrated for a good number of classes of groups.

Our conjecture is that the quasiholomorph is always a group, and aside from the known classes alluded to above, there is a great deal of computational evidence to suggest that this is true generally.

Our interest here in the quasiholomorph is in how it pertains to our discussion of isomorphic vs. non-isomorphic holomorphs and multiple holomorphs, in that the quasiholomorph gives rise to a still higher equivalence that can exist between two groups.

What is most interesting is that this equivalence exists despite the disparity in the sizes of the holomorphs (so in particular of the sizes of the respective automorphism groups).

In particular this happens for the two pairs of examples we began with:

$$C_4 \times C_4 \text{ and } C_4 \rtimes C_4$$

and

$$Q_2 \text{ and } D_4$$

Theorem

For $G_1 = C_4 \times C_4$ *and* $G_2 = C_4 \rtimes C_4$ *one has that* $\mathrm{QHol}(G_1) \cong \mathrm{QHol}(G_2)$*, and in fact, one may embed both* G_1 *and* G_2 *as subgroups of* S_{16} *so that their quasiholomorphs are equal.*

Theorem

For $G_1 = Q_2$ *and* $G_2 = D_4$ *one has that* $\mathrm{QHol}(G_1) \cong \mathrm{QHol}(G_2)$*, and in fact, one may embed both* G_1 *and* G_2 *as subgroups of* S_8 *so that their quasiholomorphs are equal.*

$C_4 \times C_4$ and $C_4 \rtimes C_4$

The equivalence $\mathrm{QHol}(G_1) \cong \mathrm{QHol}(G_2)$ is tied to a relationship that exists between $\mathcal{Q}(G_1)$ and $\mathcal{Q}(G_2)$.

For $C_4 \times C_4$ and $C_4 \rtimes C_4$ for example we have that

$$|\mathcal{Q}(C_4 \times C_4)| = 24 \text{ and } |\mathcal{Q}(C_4 \rtimes C_4)| = 72$$

which is 'counterbalanced' by the fact that

$$|\,\mathrm{Hol}(C_4 \times C_4)| = 3 \cdot |\,\mathrm{Hol}(C_4 \rtimes C_4)|$$

but more interestingly, we can basically construct the elements of $\mathcal{Q}(C_4 \times C_4)$ from those in $\mathcal{Q}(C_4 \rtimes C_4)$.

And this hearkens back to the basic structural similarity of these two groups, namely that each is generated by a pair of elements of order 4, where in the non-abelian group, one inverts the other under conjugation.

$\mathcal{Q}(C_4 \rtimes C_4) = \{N_i = \langle z_i, w_i \rangle \ i = 1 \ldots 72\}$ where $w_i z_i w_i^{-1} = z_i^{-1}$.

And as each N_i is non-abelian then $N_i^{opp} \neq N_i$ where $N_i^{opp} = Cent_B(N_i)$, and since $(N^{opp})^{opp}$ for any regular subgroup then there exists a $\sigma \in S_{72}$ such that $N_i^{opp} = N_{\sigma(i)}$

One can show that $Z(N_i) = \langle z_i^2, w_i^2 \rangle \cong C_2 \times C_2$, and that in fact all $Z(N_i)$ are identical, namely that if we identify N_1 with $\lambda(C_4 \rtimes C_4)$ then $Z(N_i) = Z(N_1)$ for all N_i.

Furthermore, under any automorphism $\alpha \in Aut(C_4 \rtimes C_4)$, one has that $\alpha(z) \in \{z, z^3, z \cdot w^2, z^3 \cdot w^2\}$ and that $z_i w_j z_i^{-1}$ and $w_i w_j w_i^{-1}$ lies in either $\langle w_j \rangle$ or $\langle z_j^2 w_j \rangle$.

So for

$$N_i = \langle z_i, w_i \rangle$$
$$N_{\sigma(i)} = \langle z_{\sigma(i)}, w_{\sigma(i)} \rangle$$

we have that $[z_i, w_{\sigma(i)}] = 1$ which means that $M_i = \langle z_i, w_{\sigma(i)} \rangle \cong C_4 \times C_4$.

Now, not all the M_i are unique, but we can show that, by the previously observed properties, all M_i are mutually normalizing, and that, if we identify M_1 with $\lambda(C_4 \times C_4)$ then $\mathcal{Q}(C_4 \times C_4)$ consists of the unique $\{M_i\}$.

D_4 and Q_2

Going back to our example of the dihedral and quaternionic groups of order 8, we observed that their holomorphs were not isomorphic, and indeed not the same size at all, namely $|Aut(D_4)| = |D_4| = 8$, while $|Aut(Q_2)| = |S_4| = 24$.

Moreover, as $\mathcal{H}(D_4) = \{\lambda(D_4), \rho(D_4)\}$ and $\mathcal{H}(Q_2) = \{\lambda(Q_2), \rho(Q_2)\}$ then their multiple holomorphs aren't isomorphic either.

However, it turns out that $\mathrm{QHol}(D_4) \cong \mathrm{QHol}(Q_2)$, and in fact, may be regarded as equal with respect to a suitable embedding of D_4 and Q_2 into S_8.

Moreover, we notice the interplay between the generators of the elements of $\mathcal{Q}(D_4)$ and $\mathcal{Q}(Q_2)$ similar to how the members of $\mathcal{Q}(C_4 \times C_4)$ were 'built' from the generators of the members of $\mathcal{Q}(C_4 \rtimes C_4)$.

We have that

$$R(Q_2,[Q_2]) = \mathcal{Q}(Q_2) = \mathcal{H}(Q_2) = \{\lambda(Q_2),\rho(Q_2)\}$$

and, from Taylor and Truman, [4], if $\lambda(Q_2) = \langle \lambda(i),\lambda(j)\rangle$ then $R(D_4,[D_4]) = \mathcal{Q}(D_4)$ consists of the following 6 groups constructed in terms of the generators of $\lambda(Q_2)$.

$$D_{i,\lambda} = \langle \lambda(i),\lambda(j)\rho(i)\rangle, \quad D_{j,\lambda} = \langle \lambda(j),\lambda(i)\rho(j)\rangle, \; D_{ij,\lambda} = \langle \lambda(ij),\lambda(i)\rho(ij)\rangle$$

$$D_{i,\rho} = \langle \rho(i),\lambda(i)\rho(j)\rangle \quad D_{j,\rho} = \langle \rho(j),\lambda(j)\rho(i)\rangle, \; D_{ij,\rho} = \langle \rho(ij),\lambda(ij)\rho(i)\rangle$$

The equivalence of these holomorphs, multiple holomorphs, and quasiholomorphs is not just an equivalence between pairs of groups, but can 'unify' three or more groups at a time.

As with some of the other groups we've seen, degree 16 is a source of several examples.

Recall that D_8 and Q_4 have isomorphic holomorphs, and they also have isomorphic multiple holomorphs and quasiholomorphs.

D_8 is a split extension of C_8 by C_2, to wit:

$$D_8 = \langle x, t \mid x^8 = 1, t^2 = 1, txt = x^{-1} \rangle$$

but there is another split extension, the quasidihedral group, determined by a different order 2 element in $Aut(C_8)$, namely

$$QD_{16} = \langle x, t \mid x^8 = 1, t^2 = 1, txt = x^3 \rangle$$

And what one has is that, while the holomorphs (resp. multiple holomorphs) of D_8 and Q_4 are, of course, isomorphic, their holomorphs (resp. multiple holomorphs) are not isomorphic to the holomorph (resp. multiple holomorph) of QD_{16}.

However $\mathrm{QHol}(D_8) \cong \mathrm{QHol}(Q_4) \cong \mathrm{QHol}(QD_{16})$ (and indeed equal under particular embeddings into S_{16}) where

$$|\mathcal{Q}(D_8)| = 8 \quad |\mathcal{Q}(Q_4)| = 8 \quad \text{and } |\mathcal{Q}(QD_{16})| = 16$$

In degree 24 we find more interesting phenomena, such as these two pairs of groups with isomorphic holomorphs:

$$\mathrm{Hol}(Q_6) \cong \mathrm{Hol}(D_{12})$$
$$\mathrm{Hol}(C_4 \times S_3) \cong \mathrm{Hol}((C_6 \times C_2) \rtimes C_2)$$

where $\mathrm{Hol}(C_4 \times S_3) \not\cong \mathrm{Hol}(Q_6)$.

What we have is a unification at the level of the multiple holomorph, with an interesting inclusion to the list:

$$\mathrm{NHol}(Q_6) \cong \mathrm{NHol}(D_{12}) \cong \mathrm{NHol}(C_4 \times S_3) \cong \mathrm{NHol}((C_6 \times C_2) \rtimes C_2)$$

where now, all of the above are isomorphic to $\mathrm{NHol}(C_2 \times (C_3 \rtimes C_4))$.

If G_1 and G_2 of the same order have isomorphic holomorphs then it follows that they have isomorphic multiple holomorphs and isomorphic quasiholomorphs.

We've seen that groups with non-isomorphic holomorphs can have isomorphic multiple holomorphs and isomorphic quasiholomorphs, and for the examples we've seen so far, if $\mathrm{NHol}(G_1) \cong \mathrm{NHol}(G_2)$ then $\mathrm{QHol}(G_1) \cong \mathrm{QHol}(G_2)$ which might make one think that the 'granularity' is non-increasing in general.

However, revisiting degree 8 we have a neat small example.

For

$$G_1 = C_4 \times C_2$$
$$G_2 = D_4$$

we have that $\mathrm{Hol}(G_1) \not\cong \mathrm{Hol}(G_2)$ (even though both have order 64) but that $\mathrm{NHol}(G_1) \cong \mathrm{NHol}(G_2)$.

However, $[\mathrm{QHol}(G_1) : \mathrm{Hol}(G_1)] = 2$ while $[\mathrm{QHol}(G_2) : \mathrm{Hol}(G_2)] = 6$ and so $\mathrm{QHol}(G_1) \not\cong \mathrm{QHol}(G_2)$.

However, one can show (again under a suitable embedding into the ambient symmetric group) that $\mathrm{QHol}(G_1) \leq \mathrm{QHol}(G_2)$.

Observations/Questions

- By the work of Mills in [2],[3] no two finite non-isomorphic groups have isomorphic holomorphs, and no finite abelian group has holomorph isomorphic to that of a non-abelian group.
- If $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ as abstract groups for groups M_1, M_2 of the same order n, does that imply that for suitably chosen regular subgroups $\hat{M}_1 \cong M_1$ and $\hat{M}_2 \cong M_2$ of S_n, we have that $\mathrm{Norm}_{S_n}(\hat{M}_1) = \mathrm{Norm}_{S_n}(\hat{M}_2)$? (basically our question from earlier)
- It seems that for two groups M_1, M_2 where $|M_1| = |M_2| = 4m$ for m odd, that $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ only for $A \cong D_{2m}$ and $B \cong Q_m$.
- It seems that for two groups M_1, M_2 of order n that one never has that $\mathrm{Hol}(M_1) \cong \mathrm{Hol}(M_2)$ unless $4 \big| n$, although one can show, for example, that $\mathrm{QHol}(C_9 \times C_3) \cong \mathrm{QHol}(C_9 \rtimes C_3)$!

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